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LETTER TO THE EDITOR

# Relation between elastic and scalar transport exponent in percolation

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**Abstract.** Using simple variational considerations, we show that the elastic critical exponent, $\tau$, is related to the conductivity index, $t$, and to the correlation length exponent, $\nu$, by $\tau \leqslant t+2\nu$. We conjecture that this relation might be an equality.

The mechanical properties of percolation structures have received considerable recent attention, experimentally—in sintered materials [1, 2], in model systems [3-6]—as well as numerically [7, 8]. By now, two bounds for the elastic critical exponent $\tau$ have been derived using the nodes-links-blobs description [9, 10]. We present here the derivation of an upper bound [11] which relates $\tau$ to the scalar transport (e.g. conductivity) index, $t$. We reproduce an argument similar to the one introduced by de Gennes [12], but we emphasise the dominant role of torques (instead of forces) at a microscopic level. A classical variational principle allows us to conclude an inequality for $\tau$. This inequality seems to be an equality in the light of the most recent and accurate simulation known to us [8].

We study the elastic behaviour of percolation lattices near threshold. Different Hamiltonians have been used to account for the elastic response of depleted media [9, 13, 14]. These can lead to very different results, each of them having its specific field of experimental relevance.

We restrict ourselves to an unambiguous case: a lattice of beams. This model, [6], shares most critical features with a network of linear and angular springs [9], but its backbone is the classical conductivity one and not the 'hairy backbone' defined by Herrmann [15], The classical theory of beams is an old and well founded framework into which any realistic elastic junction can fit, provided it satisfies elementary requirements, i.e. translational and rotational invariance.

Our $d$-dimensional lattice is constructed out of infinitely rigid nodes, having $d$ translational and $d(d-1)/2$ rotational degrees of freedom, connected by beams. These can be described by a Hamiltonian which has three contributions: a *flexure* term (and *twisting* if $d>2$) which refers to the transport of moments $M_f$ (and $M_t$), a *shear* term for the transport of forces $T$ perpendicular to the axis of the beam and an *elongation* term (transport of forces, $N$, parallel to the beam) [16]:

$$2H=\int M_f^2/(EI)\,\mathrm{d}s+\int M_t^2/(GJ)\,\mathrm{d}s+\int T^2/(GS)\,\mathrm{d}s+\int N^2/(ES)\,\mathrm{d}s. \qquad (1)$$

The integrals are taken along the beams. $E$ and $G$ are Young's and the shear moduli, $S$ the cross-sectional area of a beam and $I$ and $J$ are two moments of inertia for flexure and torsion.

We describe the percolation backbone within the nodes-links-blobs model [17-19]. According to this scheme, the structure appears as a homogeneous lattice of macro-links of size $\xi$. These macro-links are made out of blobs and singly connected bonds (see figure 1). $\xi$ diverges near the percolation threshold as $|p-p_c|^{-\nu}$. For a given boundary condition, one should expect that any macro-link is subjected to a typical vector force $F$ and a typical tensor moment $M$ whose magnitudes are related by $M \sim F\xi$.

If these macro-links were purely one-dimensional chains then, due to the divergence of $M/F$, their elastic energy $H$ would be given by the first two terms of (1)

$$H \sim M^2/K \sim F^2\xi^2/K \tag{2}$$

where $K$ is an average of the microscopic bending and twisting elastic constant, $EI$ and $GJ$, integrated along the chain. These two quantities being finite, the $K$ dependence does not introduce any critical behaviour. In a one-dimensional medium, the antisymmetric tensor $M$ remains constant when no force is applied. Component by component this equilibrium condition, for the moments, is identical to the conservation law of the current in an electrical wire. On the other hand, equilibrium of forces generally requires the existence of torques. Therefore no analogy is possible with scalar transport in this last case.

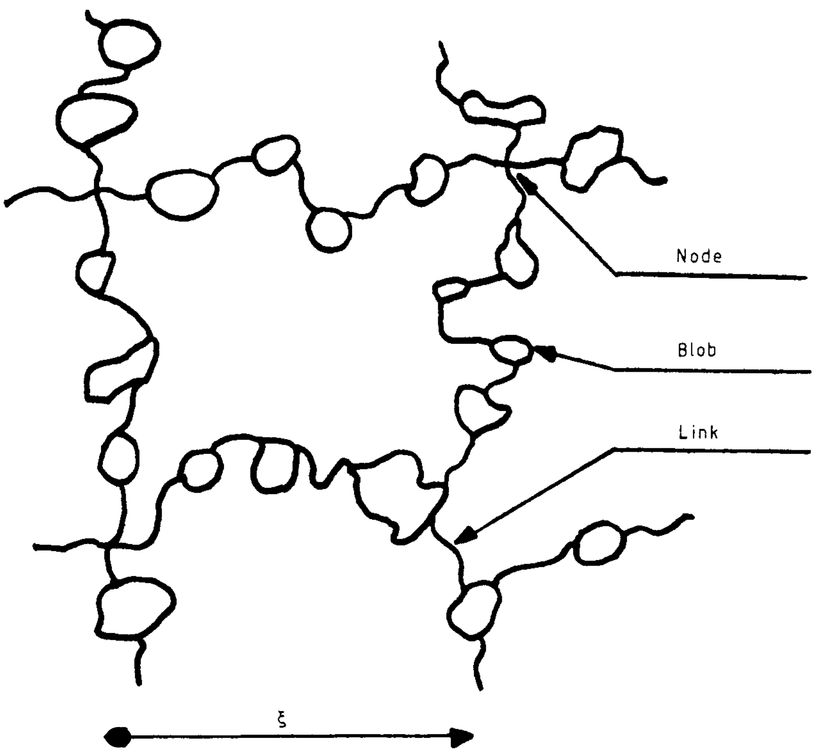


**Figure 1.** Schematic structure of the percolation backbone (nodes-links-blobs description).

However, in the usual description of macro-links, one must include blobs (i.e. loops) within the percolation backbone. Unfortunately, the existence of loops will violate the above analogy between mechanical and electrical problems. Inside a loop, forces will appear to counterbalance the torques. These forces can easily be computed by minimising the elastic energy (see the example below).

Nevertheless, one can formally write down the previous analogy:

current ↔ moments

voltage ↔ rotations

conductivity ↔ elastic modulus

dissipative energy ↔ potential energy.

Following this correspondence, we can construct a field of distributed moments, so that the macro-link is in *equilibrium*. This field can fit any torque applied at the end of the chain. Notice that this field of stress contains no force. So this will not be the actual field of stress solution of the elastic problem whenever a loop exists. We can compute the elastic energy $H_s$ of such a field. The true solution is the field of stress which minimises the elastic energy in the set of *all* equilibrium fields of stress which agree with the boundary conditions. So the true elastic energy $H_m$ satisfies

$$H_m \leqslant H_s. \tag{3}$$

$H_s$ can be written as

$$H_s \equiv M^2/2\sigma \sim F^2\xi^2/\sigma \tag{4}$$

where $\sigma$ is the conductance of the macro-link, the conductivity of which is equal to the elastic modulus relative to bending or twisting.

Now, let us define the conductance critical exponent, $\zeta$, so that

$$\sigma \sim (p-p_c)^{\zeta} \tag{5}$$

and

$$H_s \sim F^2(p-p_c)^{-\zeta-2\nu}. \tag{6}$$

We also define the elastic exponent, $\zeta'$, by

$$H_m \sim F^2(p-p_c)^{-\zeta'}. \tag{7}$$

Equation (4) holds for any macro-link. For the macroscopic solid, one has to translate $F$ into forces or displacements imposed on the boundary. Classically, in $d$ dimensions, the relation between the critical index $t$ of the macroscopic conductivity and the index $\zeta$ of the macro-link conductance is given by [18]

$$t = \zeta + \nu(d-2). \tag{8}$$

The same holds for elasticity index $\tau$ related to the macroscopic elastic modulus and $\zeta'$ which refers to a macro-link force constant

$$\tau = \zeta' + \nu(d-2). \tag{9}$$

It is important to go back from moments to forces to translate local to global properties. The reason is that, for scales larger than $\xi$, the structure is homogeneous and so no macroscopic moment can develop, whereas the opposite holds for sizes

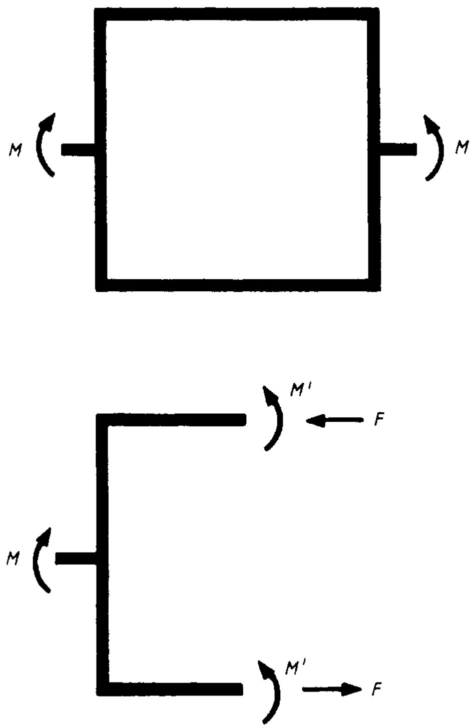


**Figure 2.** Rectangular loop (width $2a$, length $2b$) made out of beams rigidly connected with each other. If a torque $M$ is applied to it, then inside the loop a force $F$ will arise to counterbalance $M$. Equilibrium requires $M' = M/2 - Fa$. A purely scalar behaviour, as described in the text, would lead to $F = 0$ and so $M' = M/2$.

smaller than $\xi$ (topologically one-dimensional medium). Thus the inequality (3) becomes

$$\zeta' \leqslant \zeta + 2\nu \tag{10}$$

or

$$\tau \leqslant t + 2\nu. \tag{11}$$

As we have seen previously, the existence of loops prevented us from deriving an equality (or a proportionality) between $H_s$ and $H_m$. Of course, this does not mean that $\tau \neq t + 2\nu$.

For a single loop, we can pursue the calculation a little further. In the problem illustrated in figure 2, we can compute $H_s$ and $H_m$ exactly.

Let $c$ be a characteristic length of the beam used: $c = (I/S)^{1/2}$. Two dimensionless parameters are relevant now: the aspect ratio of the loop, $\alpha = b/a$ and $\beta = c/a$:

$$H_s/H_m = \frac{\beta^2\alpha(\alpha+1) + \alpha^2 + 4\alpha/3 + \frac{1}{3}}{\beta^2\alpha(\alpha+1) + \alpha/3 + \frac{1}{12}}. \tag{12}$$

The force inside the loop, $F$, amounts to

$$F = M\frac{2\alpha+1}{4\alpha\beta^2 + 4\alpha + \frac{4}{3}}. \tag{13}$$

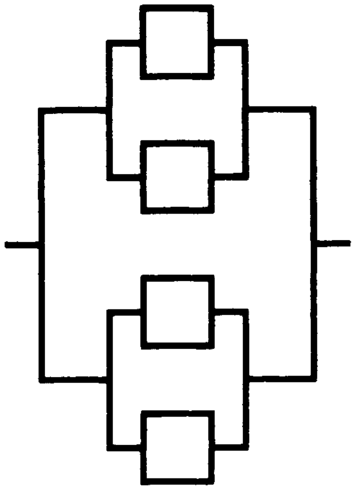

**Figure 3.** Hierarchical model of homothetic loops (here shown up to the third generation). The quantities $a$ and $b$ refer to the largest loop.

The ratio $H_s/H_m$ remains finite for any value of $\alpha$ and $\beta$, except in the limit $\alpha \to \infty$ and $\beta \to 0$. In this case

$$H_s/H_m \sim 3\alpha. \tag{14}$$

Moreover, if one considers an infinite hierarchy of loops, included inside one another, as shown in figure 3, the behaviour of $H_s/H_m$ is not qualitatively modified with respect to the previous example, i.e. $H_s/H_m$ remains finite except when $\alpha \to \infty$ and $\beta \to 0$ ((14) still holds).

In a percolation cluster, loops are not simple rectangles, and they are connected with each other in a much more complicated way than in our hierarchical model. Anyway, this last calculation shows that we may 'hope' that the qualitative behaviour of one loop may represent that of a blob. The rectangular shape of our model loop is of course schematic, but if $\alpha$ is understood as a measure of the anisotropy then we may catch the essentials of our analysis.

Above the percolation trheshold, we may expect that $\xi^{-1} < \alpha < \xi$, with an average value about 1, and $\beta_0\xi^{-1} < \beta < \beta_0$. It is then possible to encounter a few loops that make the ratio $H_s/H_m$ diverge. What we are interested in is the average of $H_s/H_m$. This means that this ratio has to be weighted by two factors: the number of loops having a given $\alpha$ and $\beta$, and the effective elastic energy of these loops. It would be of interest to study the loop size distribution and the anisotropy in the backbone blobs to make more precise determinations of the distribution of $H_s/H_m$.

Anyway, we may conjecture that the influence of the elongated loops is small enough to keep the ratio $H_s/H_m$ bounded. If this is so then the inequality $\tau \leq t + 2\nu$ becomes an equality (as suggested without derivation in [20]). The numerical simulation of [8] seems to support this conjecture: their estimate is $\tau = 3.96 \pm 0.04$ whereas $t + 2\nu = 3.97 \pm 0.01$ for $d = 2$. The equality still remains a challenge to be proven.

It is a pleasure to acknowlege numerous and fruitful discussions and helpful criticism from D Stauffer, H Herrmann, E Guyon and I Webman.

*Note added in proof.* After this letter had been submitted, we found that a conjecture similar to ours had been proposed by Sahimi [21].

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